Derivatives of Logarithmic Functions

Recall :
$$y = a^u = \frac{dy}{du} = \ln(a) \cdot a^u .$$

Q :
$$y = a^{g(x)} \Rightarrow \frac{dy}{dx} = ?$$

Chain Rule :
$$y = a^u \ , \quad u = g(x) \Rightarrow$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d}{dx}(a^u) \cdot \frac{d}{dx}(g(x))$$

$$= \ln(a)\, a^u \cdot g'(x) = \ln(a)\, a^{g(x)} g'(x)$$

Conclusion :

$$\boxed{\frac{d}{dx}\left(a^{g(x)}\right) = \ln(a)\, a^{g(x)} \cdot g'(x)}$$

Example :
$$y = \frac{2^{(3x-1)}}{7^{(x^2)}} \Rightarrow \frac{dy}{dx} = ?$$

$$y = \frac{u}{v} \ , \quad u = 2^{(3x-1)} \ , \quad v = 7^{(x^2)}$$

$$\Rightarrow u'(x) = \ln(2)\, 2^{(3x-1)} \cdot 3$$

$$\Rightarrow v'(x) = \ln(7)\, 7^{(x^2)} \cdot 2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{u'(x)\, v(x) - u(x)\, v'(x)}{(v(x))^2}$$

$$= \frac{\ln(2)\, 2^{(3x-1)} \cdot 3 \cdot 7^{(x^2)} - 2^{(3x-1)} \cdot \ln(7) \cdot 7^{(x^2)} \cdot 2x}{\left(7^{(x^2)}\right)^2}$$

Q :
$$f(x) = \log_a(x) \Rightarrow f'(x) = ?$$

Recall : $a^{\log_a(x)} = x$ ← basically the definition of $\log_a$

$f(x) = \log_a(x) \iff a^{f(x)} = x$

$\Rightarrow \frac{d}{dx}\left(a^{f(x)}\right) = \frac{d}{dx}(x)$ (Differentiate both sides)

$\Rightarrow \ln(a)\, a^{f(x)} f'(x) = 1$

$\Rightarrow f'(x) = \frac{1}{\ln(a)\, a^{f(x)}} = \frac{1}{\ln(a)\, x}$

Conclusion : $\boxed{\frac{d}{dx}(\log_a(x)) = \frac{1}{\ln(a)\, x}}$

Important Special Case : $\boxed{\frac{d}{dx}(\ln(x)) = \frac{1}{x}}$

Example

$y = \log_2(x) \cdot x^2 \Rightarrow \frac{dy}{dx} = ?$

$y = uv$, $u(x) = \log_2(x)$, $v(x) = x$

$\Rightarrow u'(x) = \frac{1}{\ln(2)\, x}$, $v'(x) = 1$

$\Rightarrow \frac{dy}{dx} = u'(x)\, v(x) + u(x)\, v'(x)$

$= \frac{1}{\ln(2)\, x} \cdot x + \log_2(x) \cdot 1$

Q/ : $y = \log_a(g(x)) \Rightarrow \frac{dy}{dx} = ?$

$$y = \log_a(u), \quad u = g(x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d}{du}\left(\log_a(u)\right) \cdot \frac{d}{dx}\left(g(x)\right)$$

$$= \frac{1}{\ln(a)\, u} \cdot g'(x) = \frac{1}{\ln(a)\, g(x)} \cdot g'(x)$$

<u>Conclusion</u> :

$$\boxed{\frac{d}{dx}\left(\log_a(g(x))\right) = \frac{g'(x)}{\ln(a)\, g(x)}}$$

<u>Important Special Case</u> :

$$\boxed{\frac{d}{dx}\left(\ln(g(x))\right) = \frac{g'(x)}{g(x)}}$$

<u>Examples</u> 1) $y = \ln(|x|) \Rightarrow \frac{dy}{dx} = ?$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \Rightarrow \ln(|x|) = \begin{cases} \ln(x) & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}, \quad \frac{d}{dx}\left(\ln(\overbrace{-x}^{g(x)})\right) = \frac{\overbrace{-1}^{=g'(x)}}{\underbrace{-x}_{=g(x)}} = \frac{1}{x}$$

$$\Rightarrow \frac{d}{dx}\left(\ln(|x|)\right) = \frac{1}{x}$$

2) $y = \log_3\left(\frac{6x+1}{7x^2+3}\right) \Rightarrow \frac{dy}{dx}$

<u>Method 1</u> $\quad y = \log_3(g(x))$ where $g(x) = \frac{6x+1}{7x^2+3}$

$\Rightarrow \quad g(x) = \dfrac{u(x)}{v(x)}$ where $u(x) = 6x+1$, $v(x) = 7x^2+3$

$\Rightarrow \quad u'(x) = 6, \quad v'(x) = 14x$

$$\Rightarrow \quad g'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2} = \frac{6(7x^2+3) - (6x+1)(14x)}{(7x^2+3)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{g'(x)}{\ln(3)\, g(x)} = \frac{\left(\dfrac{6(7x^2+3) - (6x+1)(14x)}{(7x^2+3)^2}\right)}{\ln(3) \cdot \left(\dfrac{6x+1}{7x^2+3}\right)}$$

<u>Method 2</u> $\quad y = \log_3\left(\dfrac{6x+1}{7x^2+3}\right) = \log_3(6x+1) - \log_3(7x^2+3)$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{6}{\ln(3)(6x+1)} - \frac{14x}{\ln(3)(7x^2+3)}$$

<u>Remark</u>

Compares rate of change to value of function

$$\boxed{\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)} = \text{“Relative Rate of Change”}}$$

<u>Examples</u> 1/ $P(t)$ = population size at time t.

in years

Assume $P(t) = 4000t + 6000$

$\Rightarrow$ Population is initially 6000, but grows by 4000 each year.

$$\Rightarrow \quad P'(t) = 4000$$

$$\Rightarrow \quad \frac{P'(t)}{P(t)} = \frac{4000}{4000t + 6000}$$

Note that $\lim_{t \to \infty} \frac{P'(t)}{P(t)} = 0$. This means that although the population increases by 4000 each year this represents a smaller and smaller percentage of the overall population over time.

2) $f(t) =$ GDP (gross domestic product)

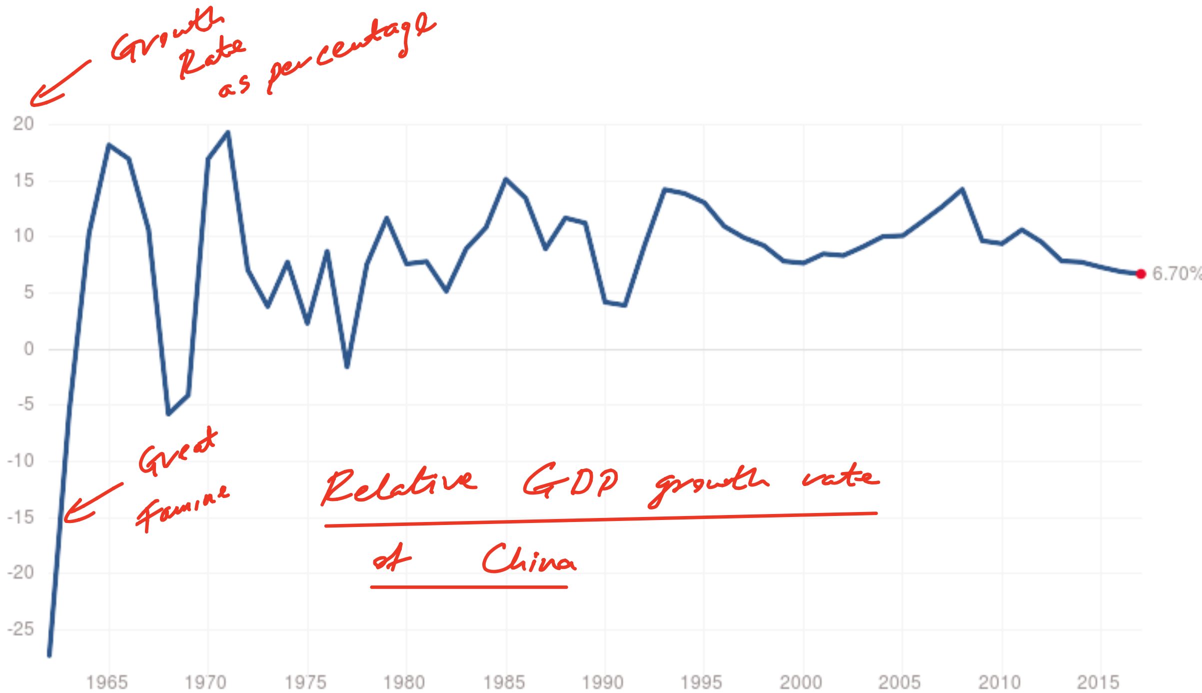
$\frac{f'(t)}{f(t)} =$ Relative growth rate of GDP at time t

Sometimes this is expressed as a percentage

ie. $100 \cdot \frac{f'(t)}{f(t)}$ %.

When someone says China has 6% GDP growth rate they mean

$$\frac{f'(t)}{f(t)} = 0.06$$


Growth Rate as percentage
20
15
10
5
0
-5
-10
-15
-20
-25
6.70%
Great Famine
Relative GDP growth rate of China
1965
1970
1975
1980
1985
1990
1995
2000
2005
2010
2015